Advanced Macroeconomics II

Assignment 1 (Suggested Answers)
(Submission Time: 3:00 pm, 11 June 2007)

1. For a neoclassical production function, show that the each factor of production earns its marginal product. Also show that if owners of capital save all their income and workers consume all their income, then the economy reaches the golden rule of capital accumulation. Explain the results.

From a firm's profit maximization problem: Max $\{L[f(k) - (r+\delta)k - w]\}$, we have

$$r + \delta = \frac{\partial F}{\partial K} = f'(k),$$

$$w = \frac{\partial F}{\partial L} = f(k) - kf'(k).$$

The above two equations state that each factor of production earns its marginal product.

If owners of capital save all their income, then

$$\dot{k} = kf'(k) - (n+\delta)k.$$

Setting $\dot{k} = 0$ gives the condition $f'(k_{gold}) = n + \delta$ that determines k_{gold}. That is, the economy reaches the golden rule of capital accumulation.

2. Consider the Solow model with the following CES production function:

$$Y = \left[(a_F K_F^{\eta} + a_I K_I^{\eta})^{\psi/\eta} + a_G K_G^{\psi}\right]^{1/\psi},$$

where Y is output; K_F is formal capital, which is subject to taxation; K_I is informal capital, which evades taxation; K_G is public capital, provided by the government and used freely by all producers; $a_i > 0$ $(i = F, I, G)$; $\eta < 1$ and $\psi < 1$. Installed formal and informal capital differ in their location and form of ownership and, therefore, in their productivity.

Output can used on a one-for-one basis for consumption or gross investment in the three types of capital. All three types of capital depreciate at the rate δ. Population is constant, and there is no technological progress.

Formal capital is subject to tax at the rate τ at the moment of its installation. Thus, the price of formal capital (in units of output) is $1 + \tau$. The price of a unit of informal capital is 1. Gross investment in public capital is the fixed fraction s_G of tax revenues. Any unused tax receipts are rebated to households in a lump-sum form. The sum of investment in the two forms of private capital is the fraction s of income net of taxes and transfers. Existing private capital can be converted on a one-for-one basis in either direction between formal and informal capital.

(a) Derive the ratio of informal to formal capital used by profit-maximizing producers.

From a firm's profit maximization: Max $\{Y - (1+\tau)(r+\delta)K_F - (r+\delta)K_I\}$, we have

$$\frac{K_I}{K_F} = \left[\frac{(1+\tau)a_I}{a_F}\right]^{\frac{1}{1-\eta}} \equiv \phi.$$

(b) In the steady state, the three forms of capital grow at the same rate. What is the ratio of output to formal capital in the steady state?

Let H_j be investment in capital j, where $j = F, I$ and G. The government's tax revenue is: $T = \tau H_F$. The law of motion for capital j is

$$\dot{K}_j = H_j - \delta K_j.$$

In the steady state, we have

$$H_i/K_i = H_j/K_j, \tag{1}$$

where $i, j = F, I$ and G. Since $H_G = s_G T = s_G \tau H_F$, equation (1) implies: $K_G/K_F = s_G \tau$. Substituting $K_I/K_F = \phi$ and $K_G/K_F = s_G\tau$ into the production function gives the ratio of output to formal capital in the steady state:

$$\frac{Y}{K_F} = \left\{[a_F + a_I \phi^{\eta}]^{\psi/\eta} + a_G(s_G\tau)^{\psi}\right\}^{\frac{1}{\psi}} \equiv \varphi. \tag{2}$$

(c) What is the steady-state growth rate of the economy?

Since $H_F + H_I = s[Y - T + (1-s_G)T] = s[Y - s_G\tau H_F$ and $H_I/H_F = K_I/K_F = \phi$, we obtain

$$H_F = \frac{sY}{1+\phi+ss_G\tau}.$$

Then the steady-state growth rate is given by

$$g = \frac{H_F}{K_F} - \delta = \frac{sY/K_F}{1+\phi+ss_G\tau} - \delta = \frac{s\varphi}{1+\phi+ss_G\tau}.$$

(d) Use numerical simulations to show that, for reasonable parameter values, the graph of the growth rate against the tax rate, τ, initially increases, then reaches a peak, and finally decreases steadily. Explain this nonmonotonic relation between the growth rate and the tax rate.

Suppose that ($s = 0.15, \delta = 0.1, s_G = 0.3, a_I = 0.7, a_G = 0.2; a_F = 0.8, \eta = 0.5, \psi = 0.8$), then we the figure below shows that the graph of the growth rate, g, against the tax rate, τ, initially increases, then reaches a peak, and finally decreases. A rise in the tax rate has two offsetting effects on the growth rate.

3. Consider the Ramsey-Cass-Koopmans model discussed in class. Suppose that

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta} \quad \text{and} \quad Y = AK^{\alpha}L^{1-\alpha}$$

where $\theta > 0$, $0 < \alpha < 1$ and $A > 0$ (constant).

(a) Set up the household's optimization problem and find the first-order conditions.

The household chooses c to maximize

$$\int_0^{\infty} e^{-\rho t} \left(\frac{c^{1-\theta} - 1}{1-\theta} \right) N dt$$

subject to its budget constraint

$$\dot{a} = w + ra - c - na.$$

The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda(w + ra - c - na).$$

The first-order conditions are

$$c^{-\theta} = \lambda$$

$$(r - n)\lambda = (\rho - n)\lambda - \dot{\lambda}$$

$$\dot{a} = a + ra - c - na$$

$$\lim_{t\to\infty} a\lambda e^{-(\rho-n)t} = 0 \quad \text{(TVC)}$$

(b) Set up the firm's problem and find the first-order conditions.

The firm chooses (K, L) to maximize $Y - (r + \delta)K - wL$ or $L[Ak^{\alpha} - (r + \delta)k - w]$, leading to the following first-order conditions

$$r + \delta = \alpha A k^{\alpha-1}$$

$$w = (1 - \alpha)Ak^{\alpha-1}.$$

(c) Find the steady-state values of k and c.

In equilibrium, $a = k$. From the above first-order conditions, we have

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - \alpha A k^{\alpha-1}$$

$$\frac{\dot{c}}{c} = (\alpha A k^{\alpha-1} - \delta - \rho)/\theta$$

$$\frac{\dot{k}}{k} = Ak^{\alpha} - c - (n+\delta)k, \quad c = \lambda^{-1/\theta}$$

Setting $\dot{\lambda} = \dot{k} = 0$ gives

$$k^* = \left(\frac{\alpha A}{\delta + \rho}\right)^{1/(1-\alpha)}$$

$$c^* = A(k^*)^{\alpha} - (n+\delta)k^*.$$

(d) Could the economy be dynamically inefficient?

First, we find the golden-rule capital stock. Choosing k to maximize $c = Ak^{\alpha} - (n+\delta)k$ gives

$$\alpha A k^{\alpha-1} = n + \delta,$$

leading to the golden-rule capital stock

$$k_{gold} = \left(\frac{\alpha A}{\delta + n}\right)^{1/(1-\alpha)}.$$

Since $\rho > n$, we have $k^* < k_{gold}$. Therefore, the economy can not be dynamically inefficient.

(e) Construct a phase diagram in (k, λ) space to show that the steady state is a saddle point. Note that λ is the co-state variable associated with the household's optimization problem.

Using the following system of differential equations to construct a phase-diagram:

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - \alpha A k^{\alpha-1},$$

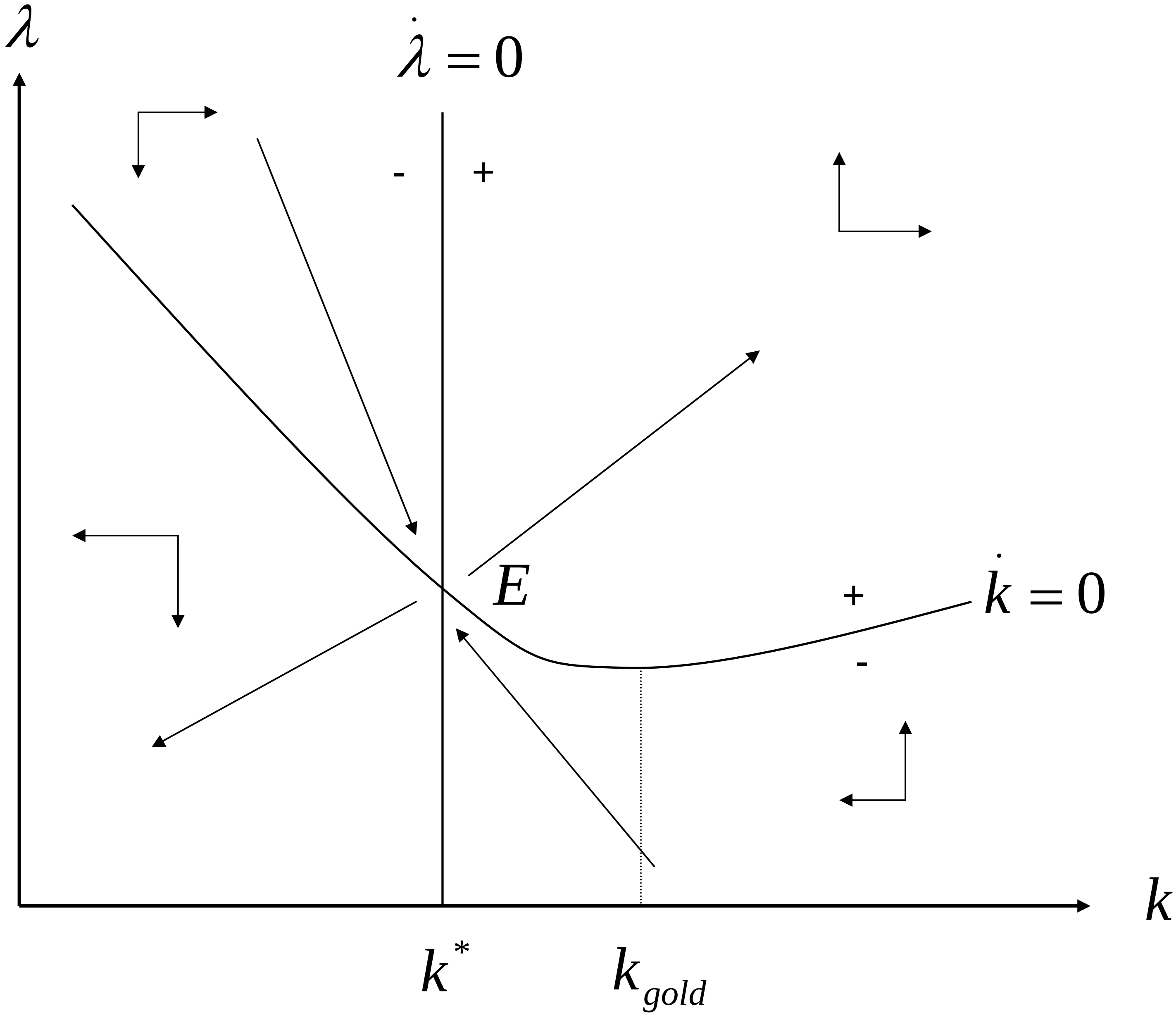
$$\frac{\dot{k}}{k} = Ak^{\alpha} - \lambda^{-1/\theta} - (n+\delta)k.$$

Setting $\dot{\lambda} = 0$ and $\dot{k} = 0$ gives respectively

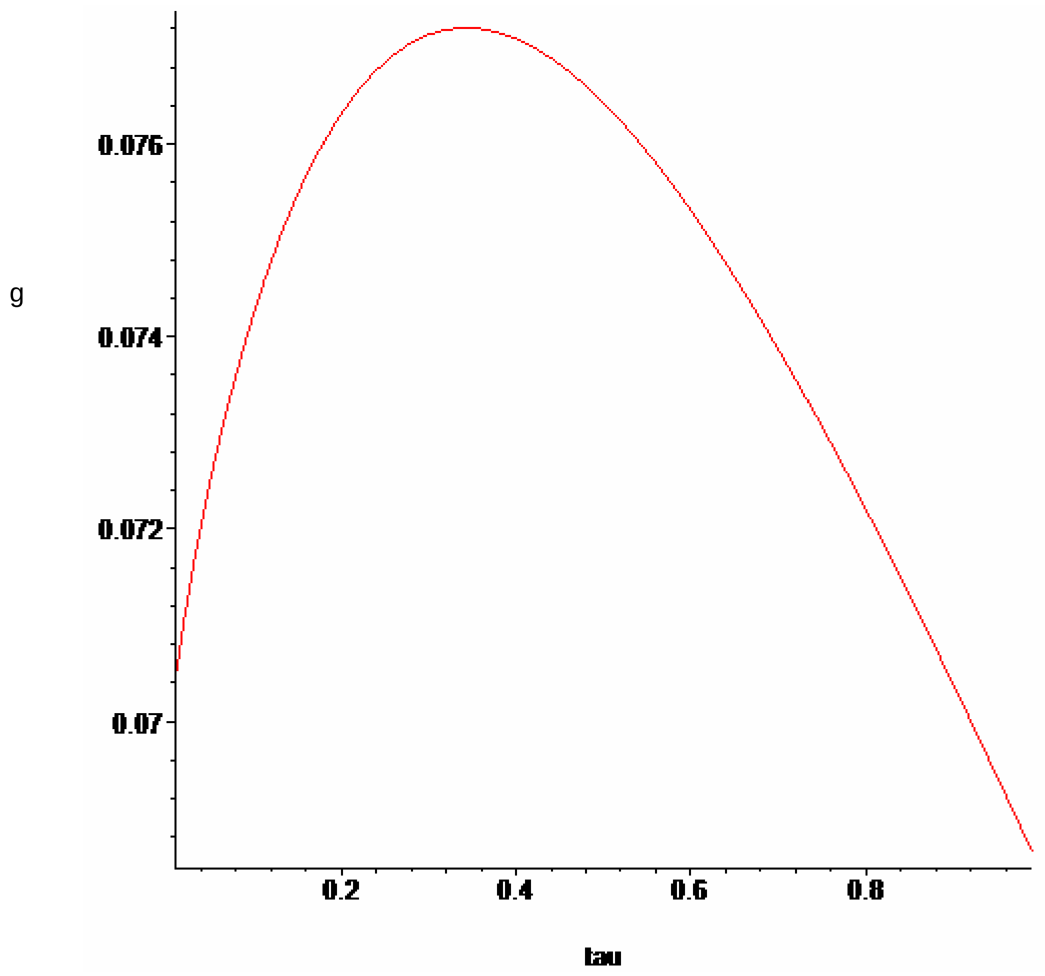
$$\lambda = [Ak^{\alpha} - (n+\delta)]^{-\theta}$$

$$k = \left(\frac{\alpha A}{\rho + \delta}\right)^{1/(1-\alpha)}.$$

Since $\partial\dot{\lambda}/\partial k = \alpha(1-\alpha)Ak^{\alpha-2} > 0$ and $\partial\dot{k}/\partial\lambda = \lambda^{-1-1/\theta}/\theta > 0$, we have the directional arrows shown in the diagram (see the Figure below). The phase-diagram shows that the steady state is a saddle point.



λ
$\dot{\lambda} = 0$
-
+
E
+
$\dot{k} = 0$
-
k
k^*
k_{gold}



g
0.076
0.074
0.072
0.07
0.2
0.4
0.6
0.8
tau