Some notes on Extensive form Representations and Subgame Perfect equilibrium

Reference These notes assume that students have a reasonably good idea about normal form representations. The students should read Chapter 1 of Gibbons to clarify thoughts on normal form games. The following notes have been written as an introduction to the general theory and should be read along with any good basic game theory text. Personally, I feel that Section 2.4 (pp 115-129) of Gibbons complement these notes nicely and I recommend strongly that you go through this section carefully.

By now, you know the *normal form representation of a game* (also called the 'strategic form') - each player simultaneously (importantly without the knowledge of other players' moves) chooses a strategy and the combination of strategies chosen by the players determines a payoff for each player. Typically, the normal form of a game is written as a payoff matrix and represents a static situation. However, all strategic interactions do not occur simultaneously. There is often some element of sequentiality in the order of moves by different players. Further, when the timing is sequential, the player moving later has information about has happened in the game before his move. All these issues of timing and information relating to sequentiality is much better captured in a game tree - this is what is also called an *extensive form representation.* Often, games involving a sequential structure are called *dynamic games.*

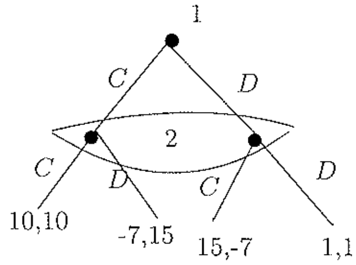
Nevertheless, it should be pointed out that we don't want to have different unconnected theories for 'normal form representation' and 'extensive form representations' of strategic situations. Ideally, every strategic interaction must have a 'normal form' as well as an 'extensive form' representation such that we can jump from one to the other with ease. Indeed the theory achieves this generalization by making some restrictions (intuitively appealing ones) on representation of information in game trees (through 'information sets') and treating the definition of 'strategy' with care. Analysis of extensive form games are made through studying allowable subtrees (called 'subgames').

1. Example of a PD

We will start developing the ideas through some simple examples. Consider the normal form of the following Prisoner's Dilemma (PD) on the left.

		Player 2	
		C	D
Player 1	C	10,10	-7,15
	D	15,-7	1,1

Normal form PD



Extensive form PD

The tree (inverted) on the right hand side above is an extensive form representation of the situation. It consists of *nodes* and branches and has the terminal payoffs at the bottom of the branches. A *node* is a decision point for a player where the player has to make a move; every node has a player in charge of the node. Above right, the filled circles at the start of every branch denote a node. The branches coming out from a node represent the different actions choices for the player who is in charge of the node. This tree captures simultaneity in the following way. Player 1 moves first (in moving first, he/she does not know what player 2 will play later) at the top node by choosing the C branch or the D branch. This is followed by two nodes (depending on player 1's choice) at each of which player 2 has to make a choice between C and D. Note that these two nodes of 2 are encapsulated in a chain (in the figure) - we call this an 'information set'- the player in charge of the nodes of an information set cannot distinguish between these nodes (here this completes the capture of simultaneity). Implicitly, if two or more nodes belong to an information set, then it is the same player who is in charge of all these nodes. Furthermore, in any preparation (strategy) to play the game the player must choose identical actions at each of these nodes - different action choices at different nodes in an information set would indicate that the player can distinguish between these nodes; we don't allow this distinction for an information set[1] . Consequently, player 2 has two pure

[1] When a player can distinguish a node perfectly (termed as perfect information), we sometimes

strategies C and D at the information set. Player 1 also has 2 strategies C and D. We formally define Information Sets.

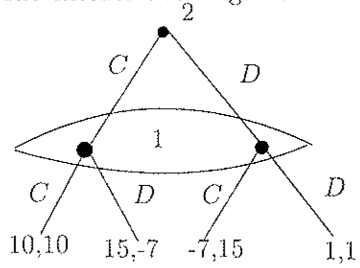
Definition 1. *An* **information set** *is a collection of decision nodes such that*

(a) the same player is in charge of all these nodes, and

(b) if the play of the game reaches a node in this collection, the player does not know which node has been reached.

Note that condition (b) implies that the player must have the same set of action choices at each node in her information set; otherwise she could distinguish (at least partially) between the nodes.

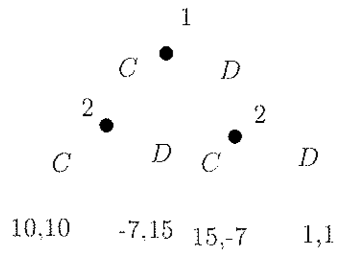
Going back to our PD example, note that an equally persuasive tree capturing all these details is the following. Player 1's payoffs are followed by player 2's payoffs in the payoff entries. This shows that the same strategic situation can sometimes be represented by more than one extensive form game.



2. Example of a Sequentially played PD

Now, we take a sequential version of the PD with player 1 moving before player 2 (technically speaking, this is no longer a PD). Before making her move player 2 knows what player 1 have done. The unique extensive form representation is given below.

call this node as a singleton (with only element) information set. Trivially, if there is only node in an information set, the player cannot confuse it with any other node in the information set.



Extensive form for sequential PD

In this extensive form, player 2 does not have an information set containing two nodes - playing second, she can distinguish between her two nodes. It is interesting how this fact translates into player 2's possible strategies.

Important In game theory, it is convenient to think that players' think about all distinguishable **contingencies** that can arise[2] in the game before actually playing the game. Next, each player submits a strategy. **A strategy for a player is a complete preparation for all distinguishable contingencies that the player may face while playing the game.** A player's strategy should prescribe for each contingency an action that the player will take if that contingency ever comes up in the course of the game's play. Once each player have submitted his/her strategy, the players take a back seat and the submitted strategies fight each other out. The interaction of the strategies produce an 'outcome path' - which tells what happens when the game is played out. To calculate the resulting payoffs when strategies interact, one has to chart the 'outcome path' that the strategies produce, and then read the terminal payoffs following this outcome path from the extensive form representation. Of course, strategies should be judiciously submitted and that is the whole point of equilibrium analysis.

Before trying an analysis on what makes judicious choice of strategies, let us tally the strategies for each player in the sequentially played PD. Player 1 has 1

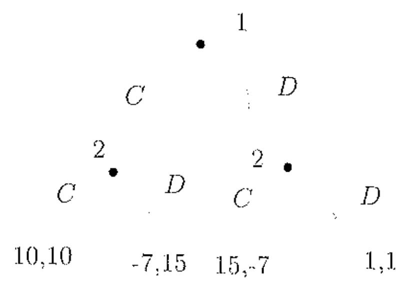
[2] As long as a player can distinguish between two different nodes, these are different contingencies. This also means that an information set with two or more nodes is one contingency.

contingency (at the top node) where he can choose any of 2 actions; so, C and D are the two possible strategies. Player 2 has 2 contingencies - the left and the right node. CD is a possible strategy for player 2 where C is what she will play at the left contingency and D is what she will play at the right contingency. Similarly, CC, DC and DD are the three other strategies that player 2 can have[3]. When strategy C of player 1 meets strategy CD of player 2, the outcome is C of player 2 followed by C of player 2 with payoffs 10,10. Similarly when strategy D of player 1 meets strategy CD of player 2, the outcome path is D of player 1 followed by D of player 2 with payoffs 1,1. The normal form of this sequential game is as follows with the only Nash equilibrium being (D, DD) with payoff 1,1.

		Player 2			
		CC	CD	DC	DD
Player 1	C	10,10	10,10	-7,15	-7,15
	D	15,-7	1,1	15,-7	1,1

However, apart from expressing a strategic situation in both an extensive form as well as a normal form representation, another important advantage in studying extensive form representations is that it allows us a much richer analytical tool than a Nash equilibrium. What strategies are judicious in the sequential version of the PD? The normal form tells us that (D, DD) looks a good analytical prescription. What would an analysis on the extensive form tell us?

[3] I am making the assumption that the first letter is the action she will play at the left node and the second letter is the action she will play at the right node.



Analysis of Extensive form of sequential PD

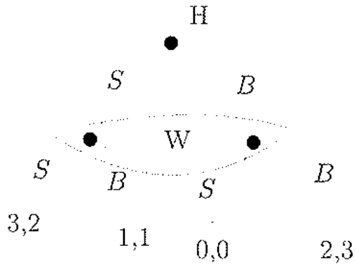
Since players think ahead rationally before playing the game, player 2 would move D if she is is ever to move in her left node (and get 15 instead of 10). Similarly, player 2 would move D if ever she had to move in her right node (and get 1 instead of -7). Both these choices are pointed with arrows. DD appears to be the logical strategy for player 2. Since player 1 is also rational, he can think ahead and decide what is best for him. If he plays C, he gets -7 (as this would be followed by D by 2) while if he plays D, he gets 1 (this would also be followed by D by 2). Player 1 should choose D. The strategy combination (D, DD) looks the logical conclusion. What we have just done is find out the *subgame perfect equilibrium* of the game (this is also the backward induction outcome). Before clarifying the meaning of the term 'subgame perfect equilibrium', let us note that the same strategy combination (D, DD) is the logical outcome (Nash equilibrium) in the normal form game. This is an accident. To see why let us consider the sequential version of the following Battle of the Sexes game.

3. Example of a Sequentially played Battle of the Sexes (BOS)

Let us write a simultaneous Battle of the Sexes game first.

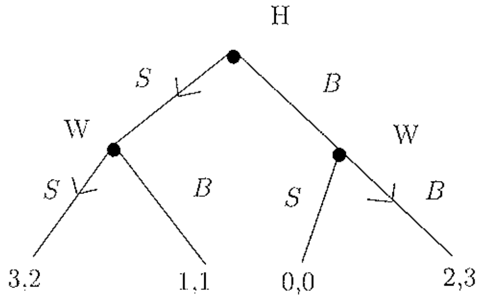
		Wife	
		S	B
Husband	S	3,2	1,1
	B	0,0	2,3

Normal form BOS



Extensive form BOS

Check that the extensive form corresponds to the normal form representation of the story. However, our focus is on the sequential version of the story where the husband moves first followed by the wife where the wife sees the husband's move. The extensive form of this modified game (with the arrows of 'looking ahead and thinking back' thrown in) is as follows.



Analysis of Exten. Form of Sequential BOS

Check why the subgame perfect strategy combination is (S, SB) (where we make the assumption that in the wife's strategy the first letter is what she does on the left node and so on). The normal form representation of this sequential game is

Wife

Husband		SS	SB	BS	BB
	S	3,2	3,2	1,1	1,1
	B	0,0	2,3	0,0	2,3

Normal form of sequential BOS

The three circled entries are the Nash equilibria. In terms of strategy configurations, they are (S, SS), (B, BB) and (S, SB). Check that the first 2 fail to satisfy the conditions of thinking ahead and looking back (e.g. in the first Nash eq. the wife is supposed to play S in the right hand node. Sequential rationality (from the extensive form game) tells us that she will never do this.

What we have shown in the above example is that the concept of Nash equilibrium is not complete in sequential games as many equilibria maybe non-credible.

It is now time to bring out the definition and purpose of subgame perfection more systematically. If players are rational (can think ahead and fathom all complexities), not only do we want players to play a best response to each other's strategies (Nash equilibrium), we would like to give players the opportunity to reevaluate their strategies at every future node that they can distinguish - this is where the strategy combination (S, SS) failed to be stable as player 2 would not play S in her right side node (a node she can distinguish). For equilibrium stability, we now insist that in any node in the game which does not belong to an information set with two or more nodes (the player has no confusion of where she is), the player's future strategy from this point on must be a best response to the truncations (from the original strategies) of other players' strategies from this point on. And this must happen for all players at all such distinguishable nodes. Rephrasing, we call the rest of the tree following a distinguishable node (singleton information set) as a 'subgame' and insist that players play a Nash Equilibrium in all such subgames. This gives us Subgame Perfect Nash Equilibrium.

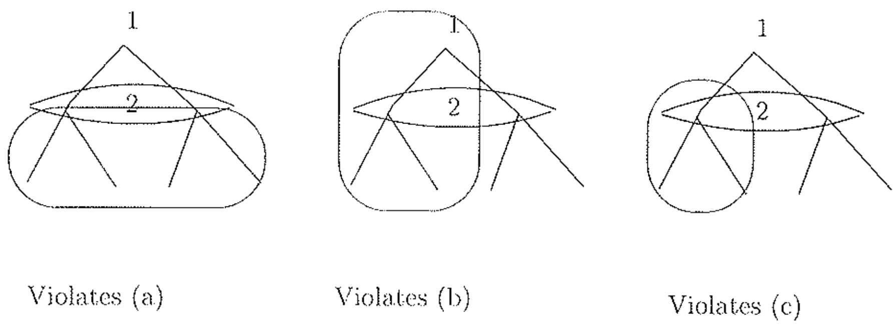
Definition 2.[4] *A* **subgame** *in an extensive form game is a sub-tree which*

(a) starts at a singleton information set,

(b) includes all decision nodes and terminal payoffs of the original tree following this node, and

(c) does not cut any information set of the extensive form representation of the original game.

Note that this requirements imply that for the extensive form representation of the simultaneous PD, none of the subtrees in ovals below (I have not labelled the actions and the terminal payoffs) are subgames.



Subtrees in ovals are not subgames

So, for simultaneous games like the PD and the original Battle of the Sexes game, the only subgame is the whole game.

Definition 3. (Selten 1965) *A strategy combination is subgame perfect Nash equilibrium if the truncations of the players' strategies constitute a Nash equilibrium for every subgame.*

[4]My definition differs a little from the definition in Gibbons - in requirement (a), he excludes the game's first decision node. I don't make this exclusion. Later, he defines a subgame perfect Nash equilibrium to be a *Nash equilibrium of the whole game* where the truncated strategies continue to be Nash equilibrium in every subgame. His insistence on starting with *a Nash equilibrium for the whole game* ensures that in the subgame (according to my definition) which starts at the first decision node, the strategies are a Nash equilibrium. Effectively, the two 'subgame perfect Nash equilibrium' definitions become identical.

So, for simultaneous games like the PD and the original Battle of the Sexes game, the only subgame perfect Nash equilibria are the Nash equilibria for the whole game.

Why does subgame perfection (defined) reduce to backward induction (thinking ahead and looking back) in finite games (games with a finite number of players which have a finite number of subgames)? The answer lies in the requirement of an equilibrium and the nature of trees and subgames. Subgames have a natural nested structure - smaller subgames sit inside larger subgames and so on. And we want a property (equilibrium in every subgame) to hold right through these nested orderings. The smallest subgames (the furthest from the start of the game) occur right at the end of these nestings with no more subgames nested in them - it is easiest to build the strategies satisfying our property from this end. So we start at the back, check strategies satisfying the property in these smallest subgames, fold these fulfilling strategies back into the next layer of subgames and so on until we reach the whole game.

For infinite games (like the infinite repetition of the the PD), there is no last subgames to start backward induction. Nevertheless, the definition of Sub game perfect Nash equilibrium continue to hold - the way to find them becomes different; we have to locate natural recursions in the problem to calculate the equilibria.

4. Existence of Subgame Perfect Equilibrium

Proposition 1. Every finite game with perfect recall has at least one subgame perfect equilibrium.

Proof. Let G be a finite game with perfect recall. Let n be the number of nodes in $T\backslash X_0$ of G. We will apply mathematical induction on n. We first consider the case with n = 1. According to Nash Theorem, G has a Nash equilibrium. Because $n = 1$, G is the only subgame of itself. Thus this NE is also an SPNE of G.

Now assume that our conclusion is true for all $n < K$. Consider any game G with $n = K$. If G has no proper subgame, then the existence of an NE implies the existence of an SPNE. Therefore we may assume that G has a proper subgame G'. If G' still has a proper subgame G", we consider G" instead of G', ..., and so on. Thus, we can finally obtain a proper subgame G` of G such that G` itself has no proper subgame. By Nash Theorem, G` has an NE in behavior strategies with some payoff vector v. We now construct a new game Γ, which is the same as G, except that G` is replaced by a terminal node with the payoff vector v. Obviously the number of non-terminal nodes in Γ is less than K. Thus by the induction assumption, Γ has an SPNE in behavior strategies. Now define the behavior strategy of every player in G by combining his choices in Γ and in G`. We will show that these strategies form an SPNE σ^* of G.

We first argue that σ^* is an NE of G. If not, then some player i can make an improvement by playing some s^i instead of σ^{*i} while all the others play their part in σ^*. Compare those components of s^i consisting the choices in G` with the corresponding components of $\sigma^{*i}$. If they are different, we can change these components of $s^i$ into those in $\sigma^{*i}$ without reducing i's payoff. (Otherwise $\sigma^*$ restricted to G` will not give an NE of G`) Therefore, without loss of generality, we may assume that $s^i$ and $\sigma^{*i}$ are different in some components not related to his choices in G`. But then, s^i restricted to Γ also give an improvement for i when it is compared with σ^{*i}. We thus have a contradiction. Therefore σ^* must be an NE of G.

To see that σ^* is subgame perfect, consider any subgame G' of G. If G` is not a subgame of G', then G' is a subgame of Γ. In this case, obviously $\sigma^*$ gives an NE of G'. Now assume that G` is a subgame of G'. Let Γ' be the new game which is the same as G' except that G` is replaced by a terminal node with the payoff vector v. Obviously σ^* gives an NE for Γ'. In an argument similar to that given in the last paragraph, one can show that σ^* induces an NE of G'.

5. A game G is said to be with perfect information, if every information set is a singleton, i.e. consisting of a single node. For a game with perfect information, the succesive process of replacing a final stage one-person subgame by a terminal node with one payoff vector corresponding to this person's optimal choice in this subgame is called backward induction. It is not difficult to establish the following

Proposition 2. A game with perfect information always has a subgame perfect equilibrium in pure strategies, and any pure strategy subgame perfect equilibrium of it can be computed by backward induction algorithm.

The proof of this proposition is similar to that of Proposition 1, and we omit it.