Advanced Macroeconomics II

Assignment 3 (Suggested Answers)
(Submission Time: 5:00 pm, 2 July 2007)

1. In the OLG model discussed in class, assume that the utility function of an individual born at time t is given by:

$$W_t = u(c_{1t}, c_{2t+1}) = (c_{1t})^\alpha + \beta\,(c_{2t+1})^\alpha\,, \quad \alpha, \beta \in (0,1). \tag{1}$$

Each individual is endowed with one unit of perishable consumption good when young and nothing when old.

(a) What are the values of c_{1t} and c_{2t+1} in the barter equilibrium?

In the barter equilibrium, there is no trade due to the absence of a double coincidence of wants. So each individual just consumes his endowment when young and nothing when old. That is, $c_{1t} = 1$ and $c_{2t+1} = 0$. [Note that $W_t = 1$.]

(b) Suppose that at time 0 the government gives to the old H_0 units of money and that the initial old and every generation thereafter believe that they will be able to exchange money for goods. What are the values of c_{1t} and c_{2t+1} in the steady-sate monetary equilibrium? Does the introduction of money improve welfare?

The optimization problem of an individual born at time t is

$$\max \ \ \{(c_{1t})^\alpha + \beta(c_{2t+1})^\alpha\}\,,$$

subject to

$$P_t(1 - c_{1t}) = M_t^d \quad \text{and} \quad P_{t+1}c_{2t+1} = M_t^d.$$

The first-order condition is:

$$\left(1 - \frac{M_t^d}{P_t}\right)^{\alpha-1} / P_t = \beta(M_t^d)^{\alpha-1}/P_{t+1}^\alpha,$$

which gives the individual's real money demand

$$\frac{M_t^d}{P_t} = \frac{P_t^\theta}{\beta^{1/(\alpha-1)}P_{t+1}^\theta + P_t^\theta} = \frac{(1+g_t)^\theta}{\beta^{1/(\alpha-1)} + (1+g_t)^\theta}, \tag{2}$$

where $\theta \equiv \alpha/(1-\alpha)$ and $1 + g_t \equiv P_t/P_{t+1}$. The money market equilibrium condition is:

$$(1+n)^t M_t^d = H_0. \tag{3}$$

Combining (2) and (3), along with g being constant in steady state, yields: $g = n$. As a result, the equilibrium values of real money balances and consumption are given by:

$$\frac{M_t^d}{P_t} = \frac{(1+n)^\theta}{\beta^{1/(\alpha-1)} + (1+n)^\theta} \equiv m \in (0,1),$$

$$c_{1t} = 1 - m,$$

$$c_{2t+1} = m(1+n).$$

Substituting $c_{1t} = 1 - m$ and $c_{2t+1} = m(1+n)$ into the utility function gives

$$W_t = (1-m)^\alpha + \beta[m(1+n)]^\alpha.$$

You need to show that $W_t > 1$, that is, the introduction of money improves welfare.

(c) Now suppose that the nominal money stock grows at a constant rate σ and that new money is introduced into the economy through interest payments to money holders. Is money superneutral in this case? Explain your answer.

In this case, the individual's optimization problem becomes

$$\max \quad \{(c_{1t})^\alpha + \beta(c_{2t+1})^\alpha\},$$

subject to

$$P_t(1 - c_{1t}) = M_t^d \quad \text{and} \quad P_{t+1}c_{2t+1} = (1+\sigma)M_t^d.$$

The first-order condition is:

$$\left(1 - \frac{M_t^d}{P_t}\right)^{\alpha-1} / P_t = \beta(1+\sigma)^\alpha (M_t^d)^{\alpha-1} / P_{t+1}^\alpha,$$

which gives the individual's real money demand

$$\frac{M_t^d}{P_t} = \frac{P_t^\theta}{\beta^{1/(\alpha-1)}(1+\sigma)^{-\theta}P_{t+1}^\theta + P_t^\theta} = \frac{(1+g_t)^\theta}{\beta^{1/(\alpha-1)}(1+\sigma)^{-\theta} + (1+g_t)^\theta}, \tag{4}$$

The money market equilibrium condition is:

$$(1+n)^t M_t^d = H_t. \tag{5}$$

In steady state, $H_t/(P_tN_t)$ is constant. Combining (4) and (5), we get:

$$1 + g = \frac{1+n}{1+\sigma}. \tag{6}$$

Inserting (6) into (4) gives the equilibrium values of real money balances:

$$\frac{M_t^d}{P_t} = \frac{(1+n)^\theta}{\beta^{1/(\alpha-1)} + (1+n)^\theta} = m.$$

As a result, we have: $c_{1t} = 1 - m$ and $c_{2t+1} = m(1+n)$, that is, money is superneutral. The reason for monetary superneutrality is that, although inflation increases, money growth does not affect the rate of return on money because the interest payment exactly compensates the money holders for the additional inflation rate.

2. Consider the Sidrauski model discussed in class, assume that the representative household's utility function is given by:

$$u(c) = \frac{(c^{\beta}m^{1-\beta})^{1-\epsilon}}{1-\epsilon}, \tag{7}$$

where $\epsilon > 0$ and $0 < \beta < 1$. The production function is:

$$Y = AK^{\alpha}N^{1-\alpha} \quad \text{or} \quad y = Ak^{\alpha}, \tag{8}$$

where $0 < \alpha < 1$ and $A > 0$. Assume that capital does not depreciate. Also assume that the money supply increases at a constant rate σ and that the money supply expands through lump-sum transfers to households.

(a) Find the first-order conditions for the representative household's optimization problems.

The representative household chooses c and m to maximize

$$\int_0^{\infty} e^{-\theta t}\frac{(c^{\beta}m^{1-\beta})^{1-\epsilon}}{1-\epsilon}dt$$

subject to the following budget constraint:

$$\dot{a} = [(r-n)a + w + x] - [c + (\pi + r)m] \tag{9}$$

and the no-Ponzi-game condition:

$$\lim_{t\to\infty} a_t exp\left\{-\left[\int_0^t (r_v - n)dv\right]\right\} = 0.$$

The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\epsilon}}{1-\epsilon} + \lambda[(r-n)a + w + x - c - (\pi + r)m].$$

The first-order conditions are (9) and

$$\beta c^{\beta(1-\epsilon)-1}m^{(1-\beta)(1-\epsilon)} - \lambda = 0 \tag{10}$$

$$(1-\beta)c^{\beta(1-\epsilon)}m^{(1-\beta)(1-\epsilon)-1} - \lambda(r+\pi) = 0 \tag{11}$$

$$\lambda(r-n) = \theta\lambda - \dot{\lambda} \tag{12}$$

$$\lim_{t\to\infty} a_t\lambda_t exp(-\theta t) = 0. \tag{13}$$

(b) Use the first-order conditions in (a) to write down the demand for real balances as a function of consumption c and the nominal interest rate i. How does the demand for real balances depends on consumption c and the nominal interest rate i?

Using (10) and (11), we have: $(1-\beta)c/(\beta m) = \pi + r = i$, which gives the demand for real balances:

$$m = \frac{(1-\beta)c}{\beta i}.$$

Obviously, the demand for real balances depends positively on consumption c and negatively on the nominal interest rate i.

(c) Set up and solve the optimization problem of the representative firm.

The representative firm chooses capital K and N to maximize its profits $(AK^{\alpha}N^{1-\alpha} - rK - wN)$, yielding the following conditions:

$$r = \alpha A k^{\alpha-1}, \tag{14}$$

$$w = (1-\alpha)Ak^{\alpha}. \tag{15}$$

(d) What are the steady-state values of inflation π^*, real money balances m^*, capital per worker k^* and consumption per worker c^*? Is money superneutral?

We can obtain these steady-state values by the following steps: (i) The assumption that the money supply expands through lump-sum transfers to households gives

$$x = \frac{\dot{M}}{PN} = \sigma m. \tag{16}$$

(ii) By definition, we have $m = M/(PN)$, leading to

$$\dot{m} = \sigma m - (\pi + n)m. \tag{17}$$

(iii) The steady-state equilibrium condition: $\dot{a} = \dot{k} = \dot{m} = \dot{\lambda} = 0$, along with (17), gives the steady-state inflation rate:

$$\pi^* = \sigma - n. \tag{18}$$

(iv) Combining (12) and (14) gives the steady-state capital stock:

$$k^* = \left(\frac{A}{\theta + n}\right)^{1/(1-\alpha)}. \tag{19}$$

(v) Using 9), (15) and (19), we get the steady-state consumption:

$$c^* = A(k^*)^{\alpha} - nk^*.$$

(vi) The real money demand function in (b) gives the steady-state real balances:

$$m^* = \frac{(1-\beta)c^*}{\beta(\sigma+\theta)},$$

where we use the following conditions: $i = r + \pi$, $r = n + \theta$ and $\pi = \sigma - n$. Since consumption and capital stock are independent of money growth, money is superneutral.

(e) What is the optimal steady-state growth rate of money σ^*? Explain.

Setting $i = 0$ (i.e., the nominal interest rate is zero), we obtain the optimal steady-state growth rate of money:

$$\sigma^* = -\theta.$$

With the optimal steady-state growth rate of money, the rate of return on money (i.e., $-\pi$) is the same as the rate of return on capital (i.e., r). Also, the marginal utility of money is zero, implying that individuals are satiated with money.

3. Consider the Seignorage-Inflation model discussed in class. Suppose that desired real money holdings are given by

$$m(t) = \frac{M(t)}{P(t)} = a - bi(t) + c\bar{Y} = C - b\pi(t), \quad b > 0,$$

where $C \equiv a + c\bar{Y} - b\bar{r}$.

(a) Suppose that the public immediately adjusts its money holdings to changes in the economic environment. What is the maximum feasible amount of seignorage S^* in the steady state?

Seignorage is given by

$$S = g_M m = g_M(C - b\pi) = g_M(C - bg_M),$$

Note that in the steady state, $\pi = g_M$. The grow rate of money g_M^* that maximizes the inflation-tax revenue S is

$$g_M^* = \frac{C}{2b}$$

and the maximum feasible amount of seignorage is

$$S^* = \frac{C^2}{4b}.$$

(b) Consider the economy in the short run. Suppose that the public adjusts its money holdings gradually toward desired holdings according to

$$\frac{\dot{m}(t)}{m(t)} = \beta[m^*(t) - m(t)], \quad 0 < \beta < 1/b. \tag{20}$$

Suppose that the government needs to finance an amount of real purchases G ($> S^*$) by printing money. Analyze the dynamics of the real money stock $m(t)$ and inflation $\pi(t)$.

Since $\dot{m}(t)/m(t) = g_M(t) - \pi(t)$ and $g_M(t)m(t) = G$, we have

$$\pi(t) = G/m(t) - \dot{m}(t)/m(t). \tag{21}$$

Substituting (21) into (20) gives

$$\frac{\dot{m}(t)}{m(t)} = \frac{b\beta}{(1-b\beta)m(t)}\left[\frac{C-m(t)}{b}m(t) - G\right]. \tag{22}$$

Since $b\beta < 1$ and $S^* = \pi m = (C-m)m/b < G$, we have

$$\frac{\dot{m}(t)}{m(t)} < 0.$$

That is, the real money stock continually falling. As a result, money growth and inflation must be continually rising (see Figure 3 in the lecture notes).

(c) Suppose that $G < S^*$ in (b). Show that inflation will not degenerate into hyperinflation.

If $G < S^*$, then $\dot{m}(t)$ is not always negative (see Figure 3). When the economy reaches the stable steady state, the real money stock is constant (i.e., $m(t) = m^*$). As a result, the inflation rate is also constant (i.e., $\pi(t) = G/m^*$). Therefore, inflation may reach a very high level, but it will not degenerate into hyperinflation.

4. Consider the Clarida-Gali-Gertler (1999) model. The model consists of the following two equations:

$$x_t = -\varphi[i_t - E_t\pi_{t+1}] + E_t x_{t+1} + g_t, \tag{23}$$

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t, \tag{24}$$

where

$$g_t = \mu g_{t-1} + \hat{g}_t, \quad \mu \in [0,1], \tag{25}$$

$$u_t = \rho u_{t-1} + \hat{u}_t, \quad \rho \in [0,1], \tag{26}$$

The policy objective function is given by:

$$\max -\frac{1}{2}E_t\left\{\sum_{i=0}^{\infty}\beta^i[\alpha x_{t+i}^2 + \pi_{t+i}^2]\right\}. \tag{27}$$

(a) Where do the two equations (23) and (24) come from?

Equation (23) comes from the household's optimal saving decisions while equation (24) comes from the firm's optimal pricing decisions.

(b) According to (23), current output depends negatively on the real interest rate and positively on expected future output. Explain why.

Current output depends negatively on the real interest rate ($i_t - E_t\pi_{t+1}$) because a rise in the real interest rate decreases current consumption (due to intertemporal substitution of consumption), leading to a decrease in current output demand.

Current output depends positively on expected future output ($E_t x_{t+1}$) because an increase in expected output raises expected consumption and thus increases current consumption (due to consumption smoothing), leading to an increase in current output demand.

(c) According to (24), inflation depends positively on the output gap and expected future inflation. Explain why.

This is because firms set nominal prices based on the expectations of future marginal costs. The output gap (x_t) captures movements in marginal costs associated with changes in current excess demand while expected inflation ($E_t\pi_{t+1}$) reflects movements in future marginal costs associated with both variation in excess demand and cost push shocks.

(d) Show that, under the optimal policy without commitment, the central bank should adjust the nominal interest rate more than one-for-one with expected future inflation.

Following section 7.3 in the lecture notes to solve the central bank's optimization problem, we obtain

$$i_t = \gamma_\pi E_t\pi_{t+1} + g_t/\varphi.$$

Since $\gamma_\pi = 1 + (1-\rho)\lambda/(\rho\alpha\varphi) > 1$, the central bank should adjust the nominal interest rate more than one-for-one with expected future inflation.

(e) Discuss the gains from commitment.

There are potential gains from commitment. First, if the central bank's target for real output exceeds the market clearing level, then an inefficiently high steady state inflation may arise in the absence of commitment (inflationary bias). In this case, the gain from commitment is to eliminate the inflationary bias. Second, even if the central bank does not have incentives to push output above potential, there may be gains from commitment: a lower effective cost (in terms of current output loss) of disinflation. This is because current price-setting depends on expectations of future economic conditions. A credible commitment to fight inflation in the future reduces the effective cost that is required to lower current inflation.