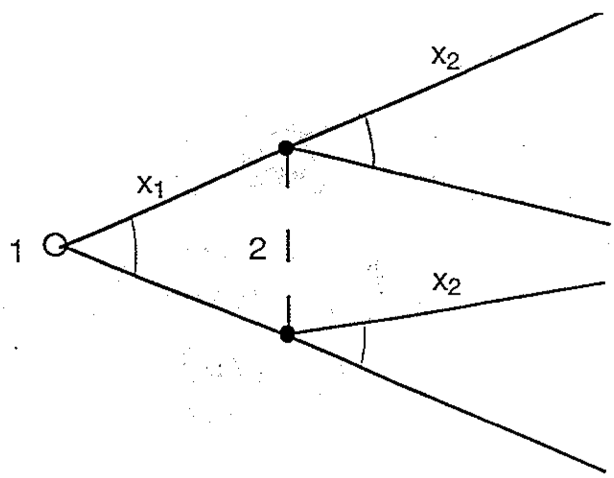
Chapter 3. The Classical Oligopoly Theory

3.1. The Cournot Competition and the Bertrand Competition

1. Consider the situation of an industry consisting of n firms. Assume that firm i produces product i, and these n products are substitutes with one another. Assume Firm i has a constant marginal cost of c_i and a fixed cost of f_i. Assume that every firm has to choose a quantity to produce simultaneously, and that the quantity produced by Firm i is denoted x_i. Assume that the market clearing price for product i is given by $p_i = \max\{0, \alpha_i - x_i - \sum_{j \neq i} \beta_{ij} x_j\}$, where α_i and β_{ij} are constants with $\alpha_i > c_i$, and $0 \leq \beta_{ij} \leq 1$. Assume that each firm wants to maximize its net profit, we then have a Cournot competition.

Obviously the story described above is an n-person game. Firm i is the ith player, and a strategy of his is a quantity to be produced. Thus i's strategy set is $\mathbf{R}_+$. Once every firm has determined his quantity, a profit can be computed for each firm as the payoff to him. Comparing this game with those we consider in the last Chapter, the main difference is that it is not a finite game but an infinite game, because the strategy set for each firm is an infinite set. In general we know that an infinite game may or may not have a Nash equilibrium. But for a Cournot competition game with linear demand functions and constant marginal costs as described above, under very general assumptions, the existence and the uniqueness of the Nash equilibrium are all guaranteed. The proof of this general argument or the computation of a Cournot equilibrium for an $n > 2$, however, requires a profound knowledge of Linear Algebra. Thus in this chapter, we only concentrate on the simple case with $n = 2$, i.e. the case of duopoly competition.

2. We now consider the duopoly quantity competition with differentiate products. Here we have the inverse demands given by $p_i = \alpha_i - x_i - \beta_{ij} x_j$, $(i, j = 1, 2; i \neq j)$. Graphically one can depict this Cournot competition game by the following game tree.



According to the discription in the last paragraph, the profit function of Firm i is given by

$$\pi_i = (\alpha_i - x_i - \beta_{ij}x_j)x_i - c_ix_i - f_i = -x_i^2 + (\alpha_i - c_i - \beta_{ij}x_j)x_i - f_i,\ (i, j = 1, 2;\ i \neq j)$$

Imagine that x_j is given and Firm i wants to choose an optimal x_i for himself. The first order condition reads

$$\partial\pi_i/\partial x_i = -2x_i + (\alpha_i - c_i - \beta_{ij}x_j) = 0,\ \text{ or }\ x_i = (\alpha_i - c_i - \beta_{ij}x_j)/2$$

The last equation is called Firm i's reaction function. It gives the best response of Firm i's because π_i is strictly concave in x_i. ($\partial^2\pi_i/\partial x_i^2 = -2 < 0$.) Let i run through 1 and 2, we have the following system of linear equations:

$$2x_1 + \beta_{12}x_2 = \alpha_1 - c_1,\quad \beta_{21}x_1 + 2x_2 = \alpha_2 - c_2$$

It is easy to solve

$$x_1 = x^*_1 = (4-\beta_{12}\beta_{21})^{-1}(2\alpha_1 - \beta_{12}\alpha_2 - 2c_1 + \beta_{12}c_2);$$

$$x_2 = x^*_2 = (4-\beta_{12}\beta_{21})^{-1}(2\alpha_2 - \beta_{21}\alpha_1 - 2c_2 + \beta_{21}c_1)$$

Note that when $(\alpha_i - c_i)/(\alpha_j - c_j) > \beta_{ij}/2$ for $i, j = 1, 2$ and $i \neq j$, x^*_1 and x^*_2 are positive quantities, they forms an Cournot (Nash) equilibrium.

3. To derive the profit formulae at the above duopoly Cournot equilibrium, we need the following

Lemma. Consider the quadratic function f defined on **R** by $f(x) = -Ax^2 + Bx + C$, where $A > 0$ and B and C are constants. Let x^* be the maximum of f. Then

$$\max f(x) = f(x^*) = Ax^{*2} + C$$

Proof. From the first order condition, one can show that $x^* = B/(2A)$, which is the global maximum of f, since f is strictly concave. Thus

$$\max f(x) = f(B/(2A)) = -A(B/(2A))^2 + B(B/(2A)) + C$$

$$= B^2/(4A) + C = A(B/(2A))^2 + C = Ax^{*2} + C$$

The lemma is thus proved.

4. According to our result in Paragragh 2 and the lemma in Paragraph 3, the duopoly

Cournot equilibrium profits are

$$\pi^*_1 = (4-\beta_{12}\beta_{21})^{-2}(2\alpha_1-\beta_{12}\alpha_2-2c_1+\beta_{12}c_2)^2 - f_1$$

$$\pi^*_2 = (4-\beta_{12}\beta_{21})^{-2}(2\alpha_2-\beta_{21}\alpha_1-2c_2+\beta_{21}c_1)^2 - f_2$$

5. To compute the equilibrium prices, one can substitute the x^*_i into the inverse demand functions. We will not derive these formulae, instead we will verify that the prices computed in this way are really positive, given the assumption that $(\alpha_i-c_i)/(\alpha_j-c_j) > \beta_{ij}/2$ for i, j = 1, 2 and $i \neq j$. In fact, because we have $x^*_1 > 0$ and $x^*_2 > 0$ under the above assumptions, we must also have $\alpha_i - x_i - \beta_{ij}x_j > 0$ for i = 1, 2. Otherwise, if, say, $\alpha_1 - x_1 - \beta_{12}x_2 \leq 0$, then the optimal quantity for Firm 1 would have to be 0!

6. Exercise. Given that $\alpha_1 = \alpha_2 = 6$, $\beta_1 = \beta_2 = 0.5$, $c_1 = 1$, $c_2 = 2$, and $f_1 = f_2 = 0$. Compute the duopoly Cournot equilibrium quantities, prices and profits by going through all the process, then verify your result with the formulae.

7. An important special case of the model described in Paragraph 2 is obtained when we have $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = 1$. Under such assumptions, the prices of the two products are always the same, and the common price p depends only on the total quantity of them. There is a total market demand function for the description of these relationship: $D(p) = \alpha - p$. We now have the duopoly quantity competion with a homogeneous product. From the formulae we have obtained, the Cournot equilibrium is given by

$$x^*_1 = 3^{-1}(\alpha - 2c_1 + c_2)\ ; \quad x^*_2 = 3^{-1}(\alpha - 2c_2 + c_1)$$

The equilibrium profits are

$$\pi^*_1 = 9^{-1}(\alpha - 2c_1 + c_2)^2 - f_1; \quad \pi^*_2 = 9^{-1}(\alpha - 2c_2 + c_1)^2 - f_2$$

(Note here we also assume that $\alpha > 2c_1 + c_2$ and $\alpha > 2c_2 + c_1$.)

8. Exercise. Given that $\alpha_1 = \alpha_2 = 6$, $\beta_1 = \beta_2 = 1$, $c_1 = 1$, $c_2 = 2$, and $f_1 = f_2 = 0$. Compute the duopoly Cournot equilibrium quantities, prices and profits.

9. Firms do not always compete with quantities, in some circumstances they may compete with prices. In everyday life one can easily observe price competition among firms. In particular, when firms cannot totally control the quantities they produce -- the telephone services for example, they usually compete with prices. When all the firms in an industry have to choose prices simultaneously and when firms' objective is for profit maximization, we have the Bertrand competition.

For simplicity here we only consider the duopoly case. Assume that Firm i produces product i and chooses a price of y_i; i = 1, 2. Assume that the quantity demanded for product is is determined by $q_i = \gamma_i - y_i + \delta_{ij}y_j$; i = 1, 2, and $i \neq j$. Here we assume that $\gamma_i > 0$, and $0 \leq \delta_{ij} \leq 1$, i.e. the two products are substitutes with each other. Assume that Firm i has a constant marginal cost of c_i and a fixed cost of f_i. Then the profit of Firm i is given by

$$\pi_i = (y_i - c_i)(\gamma_i - y_i + \delta_{ij}y_j) - f_i = -y_i^2 + (\gamma_i+c_i+\delta_{ij}y_j)y_i - c_i(\gamma_i + \delta_{ij}y_j)$$

From the first order condition one derives

$$-2y_i + (\gamma_i+c_i+\delta_{ij}y_j) = 0 \; ; \;\text{ or }\; y_i = (\gamma_i+c_i+\delta_{ij}y_j)/2$$

The last equation gives the reaction function of Firm i, and it does give the optimal price for Firm i since π_i is strictly concave in y_i. Let i run through 1 and 2, we have

$$2y_1 - \delta_{12}y_2 = \gamma_1+c_1; \quad -\delta_{21}y_1 + 2y_2 = \gamma_2+c_2$$

It is not difficult to solve

$$y^*_1 = (4-\delta_{12}\delta_{21})^{-1}[2(\gamma_1+c_1)+\delta_{12}(\gamma_2+c_2)]$$

$$y^*_2 = (4-\delta_{12}\delta_{21})^{-1}[2(\gamma_2+c_2)+\delta_{21}(\gamma_1+c_1)]$$

One can show that when $2\gamma_i+\delta_{ij}r_j > (2-\delta_{12}\delta_{21})c_i-\delta_{ij}c_j$ for i, j = 1,2 and $i \neq j$, (y^*_1,y^*_2) is a Bertrand equilibrium, at which each firm produces a positive quantity and charging a price higher than its marginal cost.

10. Exercise. Show that in duopoly Bertrand competition with a homogeneous product, when two firms all have the same constant marginal cost c and a fixed cost of 0, the only Nash equilibrium is that when each firm charges a price of c, earning a normal profit of 0.

11. Exercise. Consider the duopoly Bertrand competion with a homogeneous product. Assume that each firm has an increasing marginal cost and a zero fixed cost. What will the Bertrand equilibrium look like?

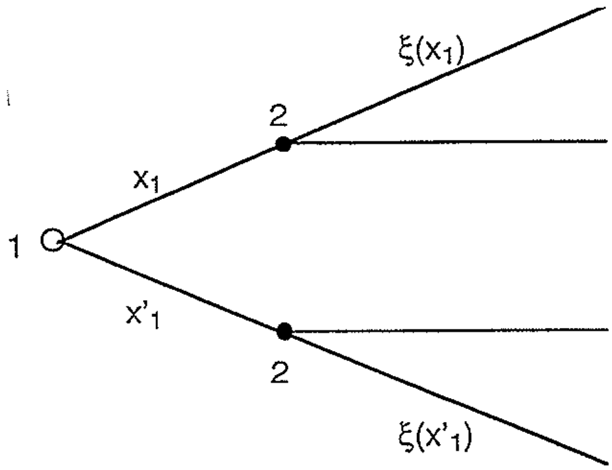
12. Exercise. Consider the following duopoly competition. Assume that the demand function of Firm i's is given by $q_i = 1 - y_i + \delta y_j$, $0 < \delta < 1$, i, j = 1, 2 and $i \neq j$. (If $1 - y_i + \delta y_j < 0$, we regard $q_i = 0$. From now on we always maintain this kind of convention.) Assume that Firm i has a fixed cost of 0, and constant marginal cost of $c < 1$, i = 1, 2. Show that quantities are always lower and prices higher in the Cournot equilibrium than in the Bertrand equilibrium. (Hint: to compute the Cournot equilibrium, you must first

solve for the inverse demands from the given demand functions, i.e. to express the y_i in terms of the q_i.)

3.2. The von Stackelberg Equilibrium and Entry Deterrence

1. Consider the following variation of the of the duopoly quantity competition model described in Exercise **3.1**-8: The total market demand and the marginal costs remain to the the same, but Firm 1 is the first mover (the leader) and Firm 2 the second mover (the follower), and Firm 2 observes Firm 1's decision before his decision is made.

Note that we have a different game! While Firm 1's strategy remains a single quantity, a strategy of Firm 2's is a function $\xi_2 : \mathbf{R}_+ \to \mathbf{R}_+$. It gives a quantity $\xi_2(x_1)$ contingent on any quantity x_1 chosen by Firm 1. The game tree looks like



A subgame perfect equilibrium of this game is called a von Stackelberg equilibrium.

Before we compute an SPE, we want to point out that, the NE we obtained in **3.1**-8 with $x^*_1 = 2$ and $x^*_2 = 1$ induces an NE for this new game: $x^*_1 = 2$, $\xi_2(x_1) = 1$ for any x_1. That means, if Firm 2 would ignore what quantity had been produced by Firm 1, and would produce a quantity of 1 in any case, then Firm 1 could do no better than producing a quantity of 2; and if Firm 1 had produced this quantity, then Firm 2's choice would have been really a best response. At this NE, the profit for Firm 1 is 4.

However, the NE described above is not subgame perfect! Look at the subgame cooresponding to the branch when a quantity of 3 has been produced by Firm 1. We will show that firm 2's best response is not a quantity of 1, but a smaller quantity. To compute 2's best response, we write down her profit function:

$$\pi_2 = (6 - 3 - x_2)x_2 - 2x_2 = -x_2^2 + x_2$$

It is easy to see that the optimal quantity for Firm 2 is 0.5 other than 1. Thus $\xi_2(3) = 1$ does not give an NE for this subgame. We have seen that the NE $x^*_1 = 2$, $\xi_2(x_1) = 1$, which is not an SPE, is generated by a incredible threat of Firm 2.

To compute an SPE, we must exclude all incredible threats of Firm 2. For this purpose we can use the backward induction algorithm. In the subgame after a quantity of x_1 is produced by Firm 1, assume that Firm 2's optimal decision is to produce $x_2 = \xi_2^*(x_1)$. As a result Firm 2 has a profit of

$$\pi_2 = (6 - x_1 - x_2)x_2 - 2x_2 = -x_2^2 + (4 - x_1)x_2$$

Thus from the first order condition and the concavity of π_2 the optimal quantity for Firm 2 is

$$\xi_2^*(x_1) = (4 - x_1)/2$$

Going backward to the first stage, Firm 1 should choose an x_1 to maximize

$$\pi_1 = [6 - x_1 - (4 - x_1)/2]x_1 - x_1 = -0.5x_1^2 + 3x_1$$

From the first order condition and the concavity of π_1 in x_1, the optimal quantity for Firm 1 is $x_1^* = 3$. Thus the unique SPE for this von Stackelberg competition game is given by $x_1^* = 3$ and $\xi_2^*(x_1) = (4 - x_1)/2$, at which Firm 1 produces 3 and Firm 2 produces 0.5. At this equilibrium Firm 1 earns a profit of $0.5(3)^2 = 4.5$. Compare this result with the Cournot competition, Firm 1 here has a higher profit. We call this as first mover's advantage.

2. Exercise. Consider the story in **3.1**-7. Now assume that Firm 1 is the first mover. Verify that at the von Stackelberg equilibrium the quantities produced are given by

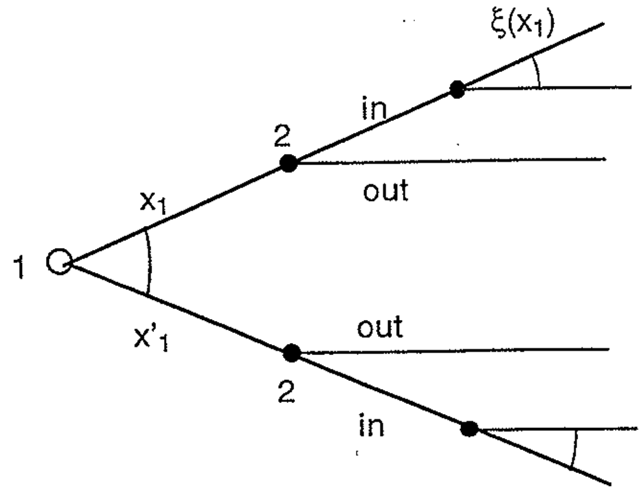
$$x_1^* = 2^{-1}(\alpha - 2c_1 + c_2), \quad x_1^* = 4^{-1}(\alpha - 3c_2 + 2c_1)$$

Can you show that in general Firm 1 has first mover's advantage?

3. Exercise. Construct an numerical example for the duopoly price competition with differentiate products and with Firm 1 as the first mover. Compare the equilibrium result with those for the Bertrand competition, and verify that Firm 2 has second mover's advantage.

4. So far we have assumed that in the oligopoly competition all the firms are existing and their fixed costs must always be paid even if they produce nothing. We now turn our attention to the new situation with potential entrants. The critical thing here is that the entrants may or may not enter into the competition, they enter if and only if they

can foresee a positive profit. Though they may have fixed costs for production, they need not to paid the costs unless they determine to enter. In the simpliest situation with one incumbent M and one entrant E, the game tree looks like



To make the situation more understandable, we assume that E will produce the same product as M. Assume the demand for this product is $D(p) = 11 - p$, where p is the price. Assume that M has a fixed cost of 4 and a marginal cost of 1, and E, if she enters, has a fixed cost of 2.25 and a marginal cost of 2.

If E did not exist, it would be easy to verify that M should produce a quantity of 5, earning a profit of 21. However, if E is waiting for entry, she will enter and produce a quantity of 2, as a result the market clearing price falls from 6 to 4, and M's profit is reduced to 11.

To compute the optimal strategy for M, assume that he produces a quantity of x_1, it is not difficult to verify that, if E does enter, her optimal quantity is $(9 - x_1)/2$, and her profit is $(9 - x_1)^2/4 - 2.25$, which is positive if and only if $9 - x_1 > 3$, i.e. $x_1 < 6$. (Note here that though we also have $(9 - x_1)^2/4 - 2.25 > 0$ if $9 - x_1 < -3$, but this never happens because Firm 1 will never produce a quantity larger than 12.) Thus the decision rule for E is: enter if and only if M produces a quantity less than 6.

According to this decision rule of E, Firm 1's profit is

$$\pi_1 = \begin{cases} [11 - x_1 - (9 - x_1)/2]x_1 - x_1 - 4\,, & x_1 \in [0, 6) \\ (11 - x_1)x_1 - x_1 - 4\,, & x_1 \in [6, \infty) \end{cases}$$

Note that there is a discontinuity point for π_1 at $x_1 = 6$. In the first interval $[0, 6)$, π_1 attains its maximum value 11.125 at $x_1 = 5.5$; in the second interval $[6, \infty)$, π_1 is decreasing, and attains its maximum value 20 at = 6. Thus the optimal strategy for Firm 1 is to produce a quantity of 6, with the entry of E being deterred.

5. Exercise. Assume that an incumbent M is producing a product, for which the market demand is $D(p) = 9 - p$, where p is the price. Assume that this incumbent has a fixed cost of 2.25 and a marginal cost of 1. Assume that a potential entrant E is waiting for entry, she enters and produces the same product, provided she foresees a positive profit. Assume that E has precisely the same technology as M. What amount should M produce?

6. Exercise. Do you think that the entry of a potential entrant can always be deterred? Explain your answer with a simple numerical example.

3.3. Repeated Play and the Folk Theoem

1. Given a finite game in normal form. Let S be the pure strategy profile set. Let Ω be the set of probability distributions over S. An element ω in Ω is called a c-mixed strategy profile. Note that the playing of a c-mixed strategy profile usually requires cooperation of the players.

Let $S = \{s, t, ..., z\}$. Let $\omega = (p_s, p_t, ..., p_z\}$ with $p_s, p_t, ..., p_z$ being the probability for s, t, ..., z, respectively, to be played. Let v(s), v(t), ..., v(z) be the payoff vectors associated to s, t, ..., z, respectively. Then the expected payoff vector associated to ω is defined by

$$u(\omega) = p_s v(s) + p_t v(t) + ... + p_z v(z)$$

As an example, in the story of the Prisoners' Dilemma (**2.2**-5), let ω be defined by [<D,D>,<C,C>; 0.5,0.5]. Then $u(\omega) = 0.5(-2,-2) + 0.5(-3,-3) = (-2.5,-2.5)$. Please note that this result is different from the expected payoff vector associated with the usual mixed strategy <(0.5,0.5),(0.5,0.5)>.

2. Exercise. Given a finite game in normal form, let Σ and Ω be the mixed strategy profile set and the c-mixed strategy profile set, respectively. Verify that for every $\sigma \in \Sigma$, there exist an $\omega \in \Omega$ which is equivalent to σ in the sense that it gives the same expected payoff vector as σ. Show that the converse need not be true.

3. To get prepared for the Folk Theorem, we re-examine the n-person finite game G with players 1, ..., I. Instead paying attention to the Nash quilibrium, we introduce a new concept. Let Σ^i be the mixed strategy set of Player i, and let Ω^{-i} be the c-mixed strategy set of the coalition -i. Define i's minimax value in this game by

$$v^i = \max_{\sigma^i \in \Sigma^i} \min_{\omega^{-i} \in \Omega^{-i}} E^i(<\sigma^i, \omega^{-i}>)$$

Note that is well-defined, because E^i is continuous on $\Sigma^i \times \Omega^{-i}$. If $E^i(<\sigma^i_d, \omega^{-i}_a>) = v^i$, we then call σ^i_d a defending strategy of i's, and ω^{-i}_a an attacking strategy of the coalition -i.

4. In the special case of a bimatrix game, we have

Proposition 1. Given a m×n bimatrix game $[A,B] = [(a_{ij},b_{ij})]_{m\times n}$. Let [A] = [A,-A] and [-B] = [-B,B] be the corresponding matrix games (or, zero-sum bimatrix games). Then at any NE of [A], the payoff to Player 1 equals to 1's minimax value in the game [A,B], and 1's strategy at this NE is a defending strategy of his for the game [A,B], and 2's strategy at this NE is an attacking strategy of hers in [A,B]; and at any NE of [-B], the payoff to 2 equals to 2's minimax value in the game [A,B], and 1's strategy at this NE is an attacking strategy of his in [A,B], and 2's strategy at this NE is a defending strategy of hers in [A,B].

Proof. By symmetry, we need only show the first half of the conclusion. Let $<\sigma, \tau>$ be an NE of [A,-A]. Because 1's payoff function in [A,B] is precisely the same as his payoff function in [A,-A], thus s is a also best response to t in the game [A,B], i.e.

$$E^1(<\sigma, \tau>) = \max_{\sigma^1\in\Sigma^1} E^1(<\sigma^1, \tau>)$$

But obviously we have for every σ^i,

$$E^1(<\sigma^1, \tau>) \geq \min_{\sigma^2\in\Sigma^2} E^1(<\sigma^1, \sigma^2>)$$

We thus have

$$E^1(<\sigma, \tau>) \geq \max_{\sigma^1\in\Sigma^1} \min_{\sigma^2\in\Sigma^2} E^1(<\sigma^1, \sigma^2>) = v^1 \qquad (1)$$

On the other hand, since t is a best response of 2 to σ in [A,-A], we have

$$E^2(<\sigma, \tau>) \geq \max_{\sigma^2\in\Sigma^2} E^2(<\sigma, \sigma^2>)$$

Note that $E^2(<\sigma^1, \sigma^2>) = -E^1(<\sigma^1, \sigma^2>)$ in [A,-A]. We then have

$$E^1(<\sigma, \tau>) \leq \min_{\sigma^2\in\Sigma^2} E^1(<\sigma, \sigma^2>)$$

But obviously we have in [A,-A] (and hence in [A,B]):

$$\min_{\sigma^2\in\Sigma^2} E^1(<\sigma, \sigma^2>) \leq \max_{\sigma^1\in\Sigma^1} \min_{\sigma^2\in\Sigma^2} E^1(<\sigma^1, \sigma^2>) = v^1$$

We thus have

$$E^1(<\sigma, \tau>) \leq v^1 \qquad (2)$$

Combining (1) and (2), we are done.

5. The following result directly follows from Proposition 1:

Corollary. All the NEs of a matrix game have the same payoff vector.

6. Exercise. Given a matrix game. Assume that $<\sigma^{*1}, \sigma^{*2}>$ is an NE. Show that

(i). Show that given any σ^2, $E^1(<\sigma^{*1}, \sigma^2>) \geq E^1(<\sigma^1, \sigma^2>)$ for $\forall\sigma^1$. (Thus σ^{*1} is an optimal strategy of 1's.

(ii). Assume that $<\tau^{*1}, \tau^{*2}>$ is another NE. Then $<\sigma^{*1}, \tau^{*2}>$ and $<\tau^{*1}, \sigma^{*2}>$ are also NEs for this game.

7. Exercise. Show that for any bimatrix game:

$$\max_{\sigma^1\in\Sigma^1} \min_{\sigma^2\in\Sigma^2} E^1(<\sigma^1, \sigma^2>) = \min_{\sigma^2\in\Sigma^2} \max_{\sigma^1\in\Sigma^1} E^1(<\sigma^1, \sigma^2>)$$

(Hint: verify that $\min_{\sigma^2\in\Sigma^2}\max_{\sigma^1\in\Sigma^1} E^1(<\sigma^1, \sigma^2>) = v^1$)

8. Given a finite game G in normal form. Let Ω be its c-mixed strategy profile set. The cooperative feasible set of G is defined by

$$CFS(G) = \{u(\omega) : \omega \in \Omega\}$$

It is easy to see that CFS(G) is the covex hull of the set of the payoff vectors associated to the pure strategy profiles.

9. Given a finite game G in normal form with player set P = {1, ..., I}. The individually rational zone is defined by

$$IRZ(G) = \{u = (u_1, ..., u_I) \in \mathbf{R}^{|P|} :\ u_i \geq v^i\}$$

where v^i is the minimax value of Player i.

The strict individually rational zone is defined by

$$SIRZ(G) = \{u = (u_1, ..., u_I) \in \mathbf{R}^{|P|} :\ u_i > v^i\}$$

10. The concepts we have just introduced are useful when we deal with the repeated play of a game G. Here we examine a simple example.

Example 1. Two firms i and j are producing products i and j, respectively. The market demands for these products are given by

$$q_1 = 2 - y_1 + 0.5\, y_2\,; \quad q_2 = 2 - y_2 + 0.5\, y_1$$

where y_i is the price of product i, q_i is the quantity of product i demanded. Assume that each firm has a constant marginal cost of 1 and a fixed cost of 0.

[Analysis]. If two firms compete for just one period, the Bertrand equilibrium can be easily calculated with the help of the formulae in **3.1**-9. Actually we have $y^*_1 = y^*_2 = 2$. Each firm produces a quantity of 1, earning a profit of 1.

On the other hand, if the two firms collude, their joint profit is

$$\Pi = (y_1 - 1)(2 - y_1 + 0.5\, y_2) + (y_2 - 1)(2 - y_2 + 0.5\, y_1)$$

$$= -y_1^2 + y_1 y_2 - y_2^2 + 2.5\, y_1 + 2.5 y_2 - 4$$

By calculation we have

$$\partial\Pi/\partial y_1 = -2y_1 + (2.5+y_2)\,; \quad \partial\Pi/\partial y_2 = -2y_2 + (2.5+y_1)$$

From the first order conditions one obtains

$$y_1 = y_2 = 2.5$$

Because $\partial^2\Pi/\partial y_1^2 = \partial^2\Pi/\partial y_2^2 = -2$, $\partial^2\Pi/\partial y_1\partial y_2 = \partial^2\Pi/\partial y_2\partial y_1 = 1$, we have $|D^2\Pi| > 0$. Thus Π is strictly concave in (y_1, y_2), and $y_1 = y_2 = 2.5$ is a global maximum. Under this situation, each firm produces a quantity of 0.75, earning a profit of 1.125, slightly higher than the Bertrand equilibrium profit.

According to the above analysis, the collusion outcome is better than the Bertrand equilibrium outcome for each firm. However, if one firm defects and charges the Bertrand equilibrium price, then the quantity demanded for the defector is 1.25, and the profit for him is 1.25. On the other hand, the quantity demanded for the other firm is 0.5, and the profit for her is only 0.75.

Assume each firm only consider the two strategies mentioned above, we then have a 2×2 bimatrix game, which looks very much like the Prisoners' Dilemma story:

		Firm 2 c	Firm 2 d
Firm 1	c	1.125, 1.125	0.75, 1.25
	d	1.25, 0.75	1, 1

where c means charging a price of 2.5, and d charging a price of 2. The only main difference between this game and the game in **2.2**-5 is that here the strategy "c" corresponds to the strategy "D" there, and "d" here to "C" there. It this game is played for only one round, since for each player "d" is the dominant strategy, the only NE is <d,d>.

However in the reality, a firm usually produces for not only one single period. It is worthwhile to examine the situation that the competition continues for, say, T periods, or, even to the infinite horizon.

When T is finite, by the backward induction method, it is not difficult to show that the only NE for the T stage game is with each firm defecting at each round. (This is left as an exercise.) The case with T being infinite is more interesting. In this case, a pure strategy of Firm i looks like $(p_1, p_2, ..., p_t, ...)$, where p_t is the pricing rule he chooses in round t, which may depend on what two firms behaved in the history from round 1 through round t-1. In particular, the following two strategies are both feasible for any firm:

A : play d in every round;

B : play c in round 1, play c in round t if the other firm plays c in every round in

the history up to round t-1, play d ever after the other firm once deviates from c.

Let ξ_t be the payoff to a firm in round t. Then this firm can choose one of the following two definitions as his payoff for the infinite horizon repeated play:

$\xi = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} \xi_t$ (the long run average payoff), or

$\xi = \sum_{t=1}^{\infty} \delta^{t-1}\xi_t$ (the long run discounting payoff)

In the discounting payoff, δ is the discounting factor, which is assumed to be a positive number less than 1.

It is not strange that we have an NE for the repeated play of the above bimatrix game when two players all adopt strategy A, no matter which definition of payoff they have chosen. This is also an NE for the repeated Bertrand competions with each firm's strategy set being [0, ∞]. What is interesting is that when two players of the repeated bimatrix game adopt the average payoff, or adopt the discounting payoff with a δ sufficiently close to 1, an NE of the repeated play of the Bertrand competitions (and hence also an NE of the repeated play of the bimatrix game) is obtained if both of them play strategy B. Here we will only examine the case with the average payoffs. The conclusion for the other case is left as an exercise.

Assume that one firm plays strategy B. We need to verify that strategy B is a best response for the other firm. In fact, by responsing with B, this firm can guarantee an long run average payoff of 1.125. Assume that she plays any other strategy B'. In case that B' leads to a sequence of constant prices: 2.5, 2.5, ..., she still gets an average payoff of 1.125. However, in case that B' leads to a deviation from the price of 2.5, say first in round t, then the payoff stream to her is

$$1.125, \ldots, 1.125, \xi_t, \xi_{t+1}, \ldots$$

where ξ_t is not greater than 1.265625 (in the one round play, the best price for her in response to a price of 2.5 of the other firm is 2.125 with the quantity demanded being 1.125); and since beginning from round t+1, her opponent always charges a price of 2, for $\tau \geq t+1$, each ξ_τ is not greater than 1. As a result, the long run average payoff to her is not greater than 1. Since in any case B' never gives an improvement to her, B is really a best response. Our conclusion is thus verified.

From the above argument, we see that, the outcome which can only be achieved through cooperation of the players can be achieved strategically in the infinite horizon repeated play.

11. In fact, for the infinite horizon repeated play, we have a very general result, which is known as the Folk Theorem:

Folk Theorem. Let G be a finite |P|-person game in normal form. Consider the infinite horizon repeated play of G. Assume that in every round, the history of the playing before this round is common knowledge to every player. Then

(i). With the long run average payoffs being adopted, every |P|-vector $v \in [CFS(G)] \cap [IRZ(G)]$ can be achieved as payoffs by an NE of the repeated play.

(ii). Given any $v \in [CFS(G)] \cap [SIRZ(G)]$, there exists a δ sufficiently close to 1, such that with the discouting payoffs being adopted, there is an NE of the repeated play achieving v as the payoffs (before discounting) in every round.

Proof. We do not give the proof in details here, because it is rather lengthy. But the basic idea is not difficult to understand: Let s be the strategy profile leading to v in the one round play. Then in the repeated play, a player's strategy is to play what he does in s in every round, so far as all the others have done the same in the history, and if, beginning from some round t, some players $i_1, ..., i_k$, (ordered lexicographically) deviate from their parts in s, then he plays his part in the attacking strategy against i_1 for the coalition $-i_1$ ever after. Under such a situation, for every player, deviating from his part from s in any round is not profitable for the infinite horizon repeated play. è

12. Exercise. Consider the price competition of n firms which are producing a homogeneous product. Assume that each firm has a fixed cost of 0 and the same constant marginal cost. Assume that the total profit for the industry in one period is 1 if these firms collude, and each firm has an equal share of the profit. If a single firm defects by charging a slightly lower price, he gets all the profit, and each of the other firm gets a profit of 0. Assume that the competition lasts to the infinite horizon, and the one period interest rate is r. Under what condition the collusion is stable?

3.4. Tutorial Problems

1. Two firms are producing the same product. The total market demand for this product is given by $D(p) = \alpha - p$. Each firm has a fixed cost of 0 and a constant marginal cost of c. In stage 1, Firm 1 determine whether to hire a profit maximizing manager or a sales revenue maximizing manager. Then in stage 2 the two firms have the duopoly Cournot competition. Assume that Firm 2's manager is a profit maximizing one, and this and the type of Firm 1's manager are common knowlege to the manager of each firm. Assume that the real objective of Firm 1's owners is profit maximization. Under what conditions, should Firm 1 hire a revenue maximing manager?

2. Consider the infinite horizon repeated play of the following bimatrix game.

		Player 2	
		c	d
Player 1	c	5, 5	-5, m
	d	m, -5	0, 0

The strategy defined below is said to be the tit-for-tat strategy:

tit-for-tat: play c in round 1, play c in round t if the other player plays c in round t-1, play d in round t if the other player plays d in round t-1.

Assume each player adopts the long run average payoff. For what values of m, an NE of the repeated play is obtained if both players play the tit-for-tat strategy?