# III Economic Growth (continued)

## D′ Endogenous Growth: Lucas' (1988) Model

### 1 Introduction

- The problem of economic development: the problem of accounting for the observed pattern, across countries and across time, in levels and rates of growth per capita income.
- Per capita income levels and growth rates are diverse. "I do not see how one can look at figures like these without seeing them as representing *possibilities*. Is there some action a government of India could take that would lead the Indian economy to grow like Indonesia's or Egypt's? If so, *what*, exactly? If not, what is it about the 'nature of India' that makes it so? The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else."
- I prefer to use the term 'theory' in a very narrow sense, to refer to an explicit dynamic system, something that can be put on a computer and *run*.

- Preferences:

$$\int_0^\infty e^{-\rho t}\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)N dt, \tag{1}$$

where $N = N_0 e^{\lambda t}$.

- Technologies:

$$Y = AK^\beta N^{1-\beta}, \tag{2}$$

where $\dot{A}/A = \mu > 0$.

- Market Clearing Condition (Resources Constraint):

$$Y = Nc + \dot{K}, \tag{3}$$

- Social Planner's Problem: The social planner's problem is to choose $K$ to maximize (1) subject to (2) and (3), where $K(0)$, $N(0)$ and $A(0)$ are given and $K(\infty) = K_\infty \geq 0$ is free. The current-value Hamiltonian is

$$\mathcal{H} = N\frac{c^{1-\sigma}-1}{1-\sigma} + \theta(Y - Nc),$$

where $Y$ is given by (2). The first-order conditions are

$$Nc^{-\sigma} - N\theta = 0, \tag{4}$$

$$Y - Nc = \dot{K}, \tag{5}$$

$$\theta\beta Y/K = \rho\theta - \dot{\theta}, \tag{6}$$

$$\lim_{t\to\infty} K\theta e^{-\rho t} = 0, \quad \lim_{t\to\infty} \mathcal{H}e^{-\rho t} = 0, \quad \lim_{t\to\infty} \theta e^{-\rho t} \geq 0, \quad \lim_{t\to\infty} K \geq 0.$$

Solving the first-order conditions:

$$(4) \Rightarrow c^{-\sigma} = \theta \Rightarrow \frac{\theta}{\theta} = -\sigma \frac{\dot{c}}{c} \tag{7}$$

$$(6) \Rightarrow \frac{\theta}{\theta} = \rho - \beta Y/K \tag{8}$$

$$(7) \quad \text{and} \quad (8) \Rightarrow g \equiv \frac{\dot{c}}{c} = \frac{\beta Y/K - \rho}{\sigma} \tag{9}$$

$$(5) \Rightarrow \frac{\dot{K}}{K} = \frac{Y}{K} - \frac{Nc}{K} = \frac{\sigma g + \beta}{\beta} - \frac{Nc}{K} \tag{10}$$

Since $Nc/K$ is constant (due to $\dot{K}/K$ and $(\sigma g + \beta)/\beta$ are constant), we have

$$\frac{\dot{K}}{K} = \frac{\dot{N}}{N} + \frac{\dot{c}}{c} = \lambda + g \tag{11}$$

$$(9) \Rightarrow \beta Y/K = \sigma g + \rho \Rightarrow \beta A K^{\beta-1} N^{1-\beta} = \sigma g + \rho \tag{12}$$

Differentiating (12) with respect $t$ gives

$$\frac{\dot{A}}{A} + (\beta - 1)\frac{\dot{K}}{K} + (1 - \beta)\frac{\dot{N}}{N} = 0$$

$$\Rightarrow \mu + (\beta - 1)(g + \lambda) + (1 - \beta)\lambda = 0$$

$$\Rightarrow g = \frac{\mu}{1 - \lambda}.$$

- Growth effects vs. level effects: The parameters $(\sigma, \rho)$ do not have *growth effects*, that is, they do not affect the growth rate

$g$. But they have *level effects*, that is, they affect the income level. The level effects can be seen from the savings rate

$$s = \frac{S}{Y} = \frac{\dot{K}}{Y} = \frac{\dot{K}/K}{Y/K} = \frac{(\lambda + g)\beta}{\sigma g + \rho}.$$

where $s$ depends on the parameters $(\sigma, \rho)$.

- Optimality of decentralized equilibrium: In this model, the decentralized equilibrium is Pareto optimal.
- The US economy [Based on Denison's (1961) study (1909-1957)]: $\lambda = 1.3\%$, $g + \lambda = 2.4\%$ or $2.9\%$ (take the average $= 2.7$), $\beta = 0.25$ and $s = 10\%$. These parameters give $g = 1.4\%$ and $\mu = 1.05\%$. The savings rate equation implies that $\rho$ and $\sigma$ satisfy

$$\rho + 0.014\sigma = 0.067.$$

Note that either output growth is underpredicted or capital growth overpredicted.

## 3 Neoclassical Growth Theory: Assessment

- Under the assumption of no factor mobility, the neoclassical growth theory is unable to account for the diversity in income levels and growth rates we observed. When factor mobility is permitted, the neoclassical theory predicts a stronger tendency to income equality and equality in growth rates.
- Variations in the parameter $(\rho, \sigma, \lambda, \beta)$ and initial technology levels $A(0)$ cannot explain why the observed income levels and growth rates are so diverse.

- Off-steady-state consideration also has difficulty explaining the large differences in income and growth:

$$g_{yt} = \beta g_{kt} + \mu.$$

- Variations in 'technology' has the potential to account for large differences in income levels and growth rates.
- The role of human capital: How does human capital accumulation affect the level and growth rate of income?

## 4 Human Capital and Growth

- Preferences: the same as (1).
- Technologies:

  (i) Final good:

  $$Y = AK^{\beta}(Nhu)^{1-\beta}h_a^{\gamma}$$

  where $A > 0$ (a constant), $0 < \beta < 1$ and $\gamma \geq 0$.

  (ii) Human capital:

  $$\dot{h} = \delta h(1-u)$$

  where $\delta > 0$.
- Market clearing condition (Resources constraint)

$$Y = Nc + \dot{K}.$$

- Decentralized equilibrium:

$$\text{Max} \quad \int_0^{\infty} e^{-\rho t}\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)N dt$$

subject to $\dot{K} = AK^{\beta}(Nhu)^{1-\beta}h_a^{\gamma}$

$$\dot{h} = \delta h(1-u).$$

The current-value Hamiltonian

$$\mathcal{H} = N\frac{c^{1-\sigma}-1}{1-\sigma} + \theta_1(Y - Nc) + \theta_2\delta(1-u)h.$$

The first-order conditions are

$$Nc^{-\sigma} - N\theta_1 = 0, \tag{13}$$

$$\theta_1(1-\beta)Y/u - \theta_2\delta h = 0 \tag{14}$$

$$\theta_1\beta Y/K = \rho\theta_1 - \dot{\theta}_1, \tag{15}$$

$$\theta_2(1-\beta)Y/h + \theta_2\delta(1-u) = \rho\theta_2 - \dot{\theta}_2, \tag{16}$$

$$Y - Nc = \dot{K}, \tag{17}$$

$$\delta h(1-u) = \dot{h}, \tag{18}$$

$$\lim_{t\to\infty}\mathcal{H}e^{-\rho t} = 0, \quad \lim_{t\to\infty}\theta_i e^{-\rho t} \geq 0, \quad i = 1, 2,$$

$$\lim_{t\to\infty} x_i \geq 0, \qquad \lim_{t\to\infty} x_i\theta_i e^{-\rho t} = 0, \quad x_i = K, h.$$

- Steady State: $g = \dot{c}/c, v = \dot{h}/h$ and $u =$ constant. Now we find the steady state growth rates. First, we find two expressions for $\dot{\theta}_1/\theta_1$.

$$(13) \Rightarrow \frac{\dot{\theta}_1}{\theta_1} = -\sigma\frac{\dot{c}}{c} = -\sigma g \tag{19}$$

$$(15) \Rightarrow \frac{\dot{\theta}_1}{\theta_1} = \rho - \beta \frac{Y}{K} \tag{20}$$

$$(19) \text{ and } (20) \Rightarrow \rho + \sigma g = \beta Y/K = \beta K^{\beta-1}(Nhu)^{1-\beta} h_a^{\gamma}. (21)$$

Differentiating (21)with respect $t$ gives (note that $h_a = h$ in equilibrium)

$$0 = (\beta - 1)(\lambda + g) + (1 - \beta)(\lambda + v) + \gamma v$$

$$0 = g(\beta - 1) + (1 - \beta + \gamma)v. \tag{22}$$

Second, we find two expressions for $\dot{\theta}_2/\theta_2$.

$$(14) \Rightarrow \frac{\theta_1}{\theta_2} = \frac{\delta hu}{(1-\beta)Y} \Rightarrow \frac{\dot{\theta}_1}{\theta_1} - \frac{\dot{\theta}_2}{\theta_2} = \frac{\dot{h}}{h} - \frac{\dot{Y}}{Y}$$

which, along with $\dot{\theta}_1/\theta_1 = -\sigma g$, implies

$$\frac{\dot{\theta}_2}{\theta_2} = (\beta - \sigma)g - (\beta - \gamma)v + \lambda. \tag{23}$$

$$(16) \Rightarrow \frac{\dot{\theta}_2}{\theta_2} = \rho - \frac{\theta_1}{\theta_2} \frac{(1-\beta)Y}{h} - \delta(1 - u) = \rho - \delta. \tag{24}$$

$$(23) \text{ and } (24) \Rightarrow (\beta - \sigma)g - (\beta - \gamma)v + \lambda = \rho - \delta. \tag{25}$$

Solving (22) and (25) gives

$$g = \frac{(\lambda + \delta - \rho)(1 - \beta + \gamma)}{\sigma(1 - \beta + \gamma) - \gamma}$$

$$v = \frac{(\lambda + \delta - \rho)(1 - \beta)}{\sigma(1 - \beta + \gamma) - \gamma}$$

- Social planner's problem: Replacing (18) by

$$\theta_1(1-\beta+\gamma)Y/h+\theta_2\delta(1-u)=\rho\theta_2-\dot{\theta}_2$$

and following exactly the same solution procedure as that for the decentralized economy, we have

$$v^*=\left[\delta-\frac{1-\beta}{1-\beta+\gamma}(\rho-\lambda)\right]/\sigma \quad \text{and} \quad g^*=\left(\frac{1-\beta+\gamma}{1-\beta}\right)v^*.$$

Note that both $v$ and $v^*$ should not exceed $\delta$, leading to the following restriction on the parameters

$$\sigma\geq 1-\left(\frac{1-\beta}{1-\beta+\gamma}\right)\left(\frac{\rho-\lambda}{\delta}\right).$$

- Decentralized equilibrium is not Pareto optimal if $\gamma>0: v*>v$. Note that if $\gamma=0$, then $v=v^*=g=g*=(\lambda+\delta-\rho)/\sigma$.
- The model's ability to fit the US data? Progress has been made in explaining cross-country differences in income levels, but more things need to be done to account for differences in growth rates.

# 5 Learning-By-Doing and Comparative Advantage

This model emphasizes the importance of on-the-job-training in the formation of human capital.

- Assume that there are two consumption goods, $c_1$ and $c_2$ and human capital is the only input in production (no physical capital). Consumption good $is$ is produced according to

$$c_i = h_i u_i N, \quad i = 1, 2, \tag{26}$$

  where $h_i$ is human capital specialized to the production of good $i$ and $u_1 (respectively, 1 - u_1)$ is the fraction of the labor force used in producing good 1 (respectively, good 2).

- Specialized human capital $h_i$ is accumulated according to

$$\dot{h}_i = \delta_i u_i h_i, \quad i = 1, 2, \tag{27}$$

  where it is assumed that good 1 is a high-technology good (i.e., $\delta_1 > \delta_2$) and that the effects of $h_i$ are entirely external in the sense that production and human capital accumulation for each good depend on the average human capital level in that industry.

- Since there is no physical capital and human capital is external, the representative consumer's optimization problem is a static problem. Assume that the consumer's utility function is given by

$$U(c_1, c_2) = [\alpha_1 c_1^{-\rho} + \alpha_2 c_2^{-\rho}]^{-1/\rho}, \tag{28}$$

  where $\alpha_i > 0, \alpha_1 + \alpha_2 = 1, \rho > -1$ and $\sigma \equiv 1/(1 + \rho)$ is the elasticity of substitution between the two goods.

## 5.1 Autarky Equilibrium

- Let $(1, q)$ be the equilibrium prices of $(c_1, c_2)$. The price of $c_2$ must equal to the marginal of substitution in consumption, i.e.,

$$q = \frac{U_2}{U_1} = \frac{\alpha_2}{\alpha_1}\left(\frac{c_2}{c_1}\right)^{-(1+\rho)}, \tag{29}$$

, which gives

$$\frac{c_2}{c_1} = \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{-\sigma}. \tag{30}$$

Profit maximization and (26) imply

$$\frac{c_2}{c_1} = \frac{u_2 h_2}{u_1 h_1}. \tag{31}$$

Combining (30) and (31), along with $u_1 + u_2 = 1$, yields

$$\frac{1 - u_1}{u_1} = \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma}\left(\frac{h_2}{h_1}\right)^{\sigma-1} = \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{1-\sigma}, \tag{32}$$

leading to

$$u_1 = \left[1 + \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{1-\sigma}\right]^{-1} - \delta_1.$$

Then from $q = h_1/h_2$, we have

$$\frac{\dot{q}}{q} = \frac{\dot{h}_1}{h_1} - \frac{\dot{h}_2}{h_2} = (\delta_1 + \delta_2)\left[1 + \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{1-\sigma}\right]^{-1} - \delta_2. \tag{33}$$

There are three cases: $\sigma > 1$, $\sigma = 1$ and $\sigma < 1$.

- Suppose that the two goods are good substitutes (i.e., $\sigma > 1$), then $\left[1 + \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{1-\sigma}\right]^{-1}$ is an increasing function of $q$. Letting $q^*$

be the solution to $\dot{q} = 0$, we have: (i) If $q(0) > q^*$, then $\dot{q} > 0$, that the economy converges to specialization in $c_1$; (ii) if $q(0) < q^*$, then $\dot{q} < 0$, that the economy converges to specialization in $c_2$; (iii) if $q(0) = q^*$, then $\dot{q} = 0$, that the economy dose not converges to specialization.

- The decentralized equilibrium is not efficient because learning-by-doing effects are external.

## 5.2 Free-Trade Equilibrium

- All countries are assumed to be small, prices in all countries will equal world prices $(1, p)$ and each country will take $p$ as given.
- All countries specialize in producing one good depending on their endowments $(h_1, h_2)$. Those with $h_1/h_2 > p$ (condition 1) produce only good 1 (so their $h_2$ endowments are fixed) while those with $h_1/h_2 < p$ (condition 2) produce only good 2 (so their $h_1$ endowments remains constant). As a result, the world supply of good $i$ is given by

  $$c_i = \sum_{\text{condition } i} h_i,$$

  which gives

  $$\frac{c_2}{c_1} = \frac{\Sigma_{\text{condition } 2} h_2}{\Sigma_{\text{condition } 1} h_1}.$$

- Since all counties have identical homothetic preferences, world relative demand remains the same as in the autarky case, i.e.,

  $$\frac{c_2}{c_1} = \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} p^{-\sigma}$$

- Dynamics of $p$: From

$$\frac{c_2}{c_1} = \left(\frac{\alpha_1}{\alpha_1}\right)^{\sigma} p^{-\sigma} = \frac{\Sigma_{\text{condition 2}} h_2}{\Sigma_{\text{condition 1}} h_1},$$

we have

$$\frac{\dot{p}}{p} = \frac{\delta_1 - \delta_2}{\sigma} > 0, \tag{34}$$

if producers do not switch from one good to the other.

- Possibility of switching production: (i) Good 2 producers do not switch because if $h_1(0)/h_2(0) < p(0)$, then $h_1(t)/h_2(t) < p(t), \quad \forall t > 0$; (ii) good 1 producers switch if $h_1(0)/h_2(0) > p(0)$ and $h_1(t)/h_2(t) < p(t), \qquad \forall t > t^*$. This can occur only if $\dot{p}/p > \dot{h}_1/h_1$ which is equivalent to $\sigma < 1 - \delta_2/\delta_1$.

- Growth rates real output: Assume that $\sigma \geq 1 - \delta_2/\delta_1$, then (34) holds true. As a result, good 1 producers's growth rate is

$$g_1 = \delta_1$$

and good 2 producers' growth rate is

$$g_2 = \delta_2 + \dot{p}/p = \delta_2 + \frac{\delta_1 - \delta_2}{\sigma}.$$

We can easily see that $g_1 > g_2$ if $\sigma > 1$. That is, if the two goods are good substitutes, then countries producing the high-learning good (good 1) have a higher growth rate.

- The model generates endogenous growth. Different countries have different growth rates.

- Important factors the model does not capture: composition of demand and introduction of new goods.

- Strategies for economic development: import substitution (depending on initial comparative advantage) and export promotion (through taxes and subsidies).